Wave Propagation in a Moving Cold Magnetized Plasma

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Polarization relations and dispersion equations are derived for media electrically anisotropic in the comoving frame. The obtained results are discussed mainly for cold magnetized plasmas, briefly for uniaxial dielectric crystals. Special directions of wave propagation are considered.

1. Introduction

In a previous paper [1] dispersion equations and polarization relations were derived for media anisotropic in the comoving frame and expressed by quadratic and bilinear forms. These expressions are in contrast to the conventional representations by means of determinants and subdeterminants. While the quadratic and bilinear forms are Lorentz covariant, the matrix-elements for the determinants and subdeterminants lack this important property in general. To assure the Lorentz covariance of the conventional method the calculation of the matrix elements becomes very involved. This can be avoided using the different approach of the previous paper [1].

To illustrate this procedure it will be applied in the present paper to media electrically anisotropic in the comoving frame, i.e. dielectric crystals and, especially, cold magnetized plasmas (without spatial dispersion). An isotropic magnetic permeability shall, however, be retained in the comoving frame to include weakly magnetic crystals.

For media anisotropic in the comoving frame the three- and four-dimensional material tensors can be represented by three- and four-dimensional projectors, respectively. For media electrically anisotropic in the comoving frame the same projectors can be used to represent the tensors which built up the quadratic and bilinear forms. Therefore the latter can be easily evaluated for these media.

The three-dimensional calculations for media at rest in Sect. 2 recover the known dispersion equations, i.e. Aström's dispersion equation for magnetized cold plasmas and Fresnel's wave normal equation for uniaxial crystals. An analogous four-dimensional calculation yields the generalization to moving media (Section 3). The dispersion equations so obtained for moving gyrotropic media are then discussed qualitatively for various special media and special directions of wave propagation (Sections 4, 5). Finally, in the last section, the polarization relations are specialized to media gyrotropic in the comoving frame.

Throughout this paper three-dimensional vectors and tensors are written in symbolic notation (with $\mathbf{I}$ as unit tensor), four-dimensional vectors and tensors in index notation with Greek indices running from 0 to 3. Dashed indices are merely labels counting projectors and eigenvalues. Quantities in the comoving frame are denoted by dashs. SI units are used and a flat space is assumed.

2. Dispersion equations for gyrotropic media at rest

In this section we shall specialize the three-dimensional dispersion equations for generally anisotropic media, which we derived in a previous work [1] to gyrotropic media. To do this we first rewrite the obtained results. In the case of a medium at rest (moving media will be treated in the next section) we obtained the two equivalent dispersion equations ([1], Eq. (4.6))

$$\boldsymbol{n}\cdot\mathbf{A}\cdot\boldsymbol{n}-\det\mathbf{C}=0 \tag{2.1}$$

and ([1], Eq. (4.10))

$$\boldsymbol{n}\cdot\hat{\boldsymbol{\epsilon}}\cdot\mathbf{A}\cdot\boldsymbol{n}=0\,, \tag{2.2}$$

with the refractive index vector $\boldsymbol{n}$ defined by

$$\boldsymbol{n}=\frac{c}{\omega}\,\boldsymbol{k}=\frac{1}{\omega\sqrt{\varepsilon_0\mu_0}}\,\boldsymbol{k}\,.$$

The general definition of the tensor $\mathbf{C}$ is given in [1] Equation (4.1b). The tensor $\mathbf{A}$ is the adjoint of $\mathbf{C}$, defined by

$$\mathbf{C}\cdot\mathbf{A}=\mathbf{A}\cdot\mathbf{C}=\mathbf{I}\det\mathbf{C}\,, \tag{2.3}$$

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which exists even for singular tensors $\mathbf{C}$ with vanishing determinant.

In the following we will restrict our calculations to electrically anisotropic media, i.e. dielectric crystals and plasmas. The magnetic properties of the medium are represented by a scalar permeability $\mu = 1/\varkappa$ to include weakly magnetic crystals ([2], Section 14.1). In this case the tensor $\mathbf{C}$ reduces to

$$\mathbf{C} = n^2\,\mathbf{I} - \hat{\boldsymbol{\epsilon}}\,. \tag{2.4}$$

The effective dielectric tensor $\hat{\boldsymbol{\epsilon}}$ is defined by ([1], Eqs. (3.3b), (2.4))

$$\hat{\boldsymbol{\epsilon}} := \frac{1}{\varepsilon_0\,\mu_0\,\varkappa}\left(\boldsymbol{\epsilon} + \frac{i}{\omega}\,\boldsymbol{\sigma}\right). \tag{2.5}$$

In the case of a gyrotropic medium, where the anisotropy is caused by an axial vector, the dielectric tensor $\boldsymbol{\epsilon}$ and the conductivity tensor $\boldsymbol{\sigma}$ are axial tensors ([3], Appendix B). We assume, that the anisotropy for both tensors is caused by parallel axial vectors. Hence $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ have the same symmetry axis. Thus the effective dielectric tensor $\hat{\boldsymbol{\epsilon}}$ and subsequently $\mathbf{C}$ and $\mathbf{A}$ have the same symmetry axis as $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$. They are axial tensors, too.

Specializing the dispersion equations (2.1), (2.2) to gyrotropic media requires mathematically to invert and multiply axial tensors. A suitable tool for this purpose is the representation of axial tensors by a complete set of three orthogonal projectors [4]. Then the effective dielectric tensor $\hat{\boldsymbol{\epsilon}}$ is represented in a "diagonal form" by

$$\hat{\boldsymbol{\epsilon}} = \hat{\varepsilon}_0\,{}_0\mathbf{P} + \hat{\varepsilon}_{+1}\,{}_{+1}\mathbf{P} + \hat{\varepsilon}_{-1}\,{}_{-1}\mathbf{P} \tag{2.6}$$

with the (hermitian) orthogonal projectors defined by ([4], Eqs. (9))

$$\begin{aligned} {}_0\mathbf{P} &:= \hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}}\,, \\ {}_{\pm1}\mathbf{P} &:= \tfrac{1}{2}\,(\mathbf{I} - \hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}} \pm i\,\hat{\boldsymbol{B}} \times \mathbf{I}) \end{aligned} \tag{2.7}$$

and $\hat{\boldsymbol{B}}$ as the unit axial vector.

Instead of the eigenvalues $\hat{\varepsilon}_{\pm1}$ their linear combinations

$$\hat{\varepsilon}_{\pm} := \tfrac{1}{2}\,(\hat{\varepsilon}_{+1} \pm \hat{\varepsilon}_{-1}) \tag{2.8}$$

as the coefficients of the (hermitian) tensors

$$\begin{aligned} {}_{+1}\mathbf{P} + {}_{-1}\mathbf{P} &= \mathbf{I} - \hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}} = 2\,\mathrm{Re}\,{}_1\mathbf{P} = 2\,\mathrm{Re}\,{}_{-1}\mathbf{P}, \\ {}_{+1}\mathbf{P} - {}_{-1}\mathbf{P} &= i\,\hat{\mathbf{B}} \times \mathbf{I} \;= 2\,i\,\mathrm{Im}\,{}_1\mathbf{P} = -\,2\,i\,\mathrm{Im}\,{}_{-1}\mathbf{P} \end{aligned} \tag{2.9}$$

are used in the "eigenvalue representation" ([5], Eq. (1.11))

$$\hat{\boldsymbol{\epsilon}} = \hat{\varepsilon}_0\,{}_0\mathbf{P} + \hat{\varepsilon}_+\,2\,\mathrm{Re}\,{}_1\mathbf{P} + \hat{\varepsilon}_-\,2\,i\,\mathrm{Im}\,{}_1\mathbf{P}\,. \tag{2.10}$$

Since the projector $2\,\mathrm{Re}\,{}_1\mathbf{P} = 2\,\mathrm{Re}\,{}_{-1}\mathbf{P}$ has an obvious physical meaning in contrast to ${}_{\pm1}\mathbf{P}$, the eigenvalue representation (2.10) is often more convenient than the diagonal representation (2.6).

The gyrotropic medium is completely characterized by the projectors (2.7) and the three eigenvalues $\hat{\varepsilon}_0$, $\hat{\varepsilon}_{\pm1}$ of the effective dielectric tensor $\hat{\boldsymbol{\epsilon}}$. These eigenvalues themselves are determined by the eigenvalues $\varepsilon_{a'}$, $\sigma_{a'}$ of the dielectric tensor $\boldsymbol{\epsilon}$ and the conductivity tensor $\boldsymbol{\sigma}$, respectively, in the form

$$\hat{\varepsilon}_{a'} = \frac{1}{\varepsilon_0\,\mu_0\,\varkappa}\left(\varepsilon_{a'} + \frac{i}{\omega}\,\sigma_{a'}\right), \quad a' = 0, \pm 1\,. \tag{2.11}$$

In the case of a magnetized plasma ($\boldsymbol{\epsilon} = \varepsilon_0\,\mathbf{I}$, $\varkappa = 1/\mu_0$) the eigenvalues $\hat{\varepsilon}_{a'}$ are directly determined by the eigenvalues $\sigma_{a'}$ of the conductivity tensor and Eqs. (2.11) reduce to

$$\hat{\varepsilon}_{a'} = 1 + i\,\sigma_{a'}/\varepsilon_0\,\omega\,, \quad a' = 0, \pm 1\,. \tag{2.12}$$

In the special case of a cold electron plasma the eigenvalues of the conductivity tensor are given by ([5], Eqs. (1.12) with (1.9), (1.7))

$$\sigma_{a'} = \frac{\omega_\mathrm{p}{}^2\,\varepsilon_0}{-\,i\,\omega + \nu + i\,a'\,\omega_\mathrm{B}}\,, \quad a' = 0, \pm 1 \tag{2.13}$$

with ν as the collision frequency, $\omega_\mathrm{p}/2\pi$ as the plasma frequency and $\omega_\mathrm{B}/2\pi$ as the cyclotron frequency.

For a dielectric crystal ($\boldsymbol{\sigma} \equiv 0$) with principal axes $\hat{\boldsymbol{P}}_{0,\pm1}$ the projectors are

$${}_{a'}\mathbf{P} := \hat{\boldsymbol{P}}_{a'}\,\hat{\boldsymbol{P}}_{a'} \tag{2.14a}$$

and the eigenvalues of the effective dielectric tensor $\hat{\boldsymbol{\epsilon}}$ are given by

$$\hat{\varepsilon}_{a'} = \varepsilon_{a'}/\varepsilon_0\,\mu_0\,\varkappa\,, \quad a' = 0, \pm 1 \tag{2.14b}$$

which for $\varepsilon_{+1} = \varepsilon_{-1}$ include, as a special case, an uniaxial crystal with $\hat{\boldsymbol{P}}_0$ as symmetry axis.

The orthogonality relations between the projectors (2.7) or (2.14a) can be used to relate the eigenvalues $\hat{\varepsilon}_{a'}$ to those of the tensors $\mathbf{C}$ (2.4) and $\mathbf{A}$ (2.3). One obtains

$$C_{a'} = n^2 - \hat{\varepsilon}_{a'}\,, \quad a' = 0, \pm 1\,, \tag{2.15a}$$

$$\begin{aligned} C_+ &= \tfrac{1}{2}\,(C_{+1} + C_{-1}) = n^2 - \hat{\varepsilon}_+\,, \\ C_- &= \tfrac{1}{2}\,(C_{+1} - C_{-1}) = -\,\hat{\varepsilon}_- \end{aligned} \tag{2.15b}$$

and

$$A_{a'} = C_{b'}\,C_{c'} = (n^2 - \hat{\varepsilon}_{b'})(n^2 - \hat{\varepsilon}_{c'})\,,$$
$$a', b', c' = 0, \pm 1; \quad a' \neq b' \neq c'\,, \quad (2.16\text{a})$$

$$A_+ = \tfrac{1}{2}(A_{+1} + A_{-1}) = C_0\,C_+ = (n^2 - \hat{\varepsilon}_0)(n^2 - \hat{\varepsilon}_+)\,, \quad (2.16\text{b})$$

$$A_- = \tfrac{1}{2}(A_{+1} - A_{-1}) = -\,C_0\,C_- = (n^2 - \hat{\varepsilon}_0)\,\hat{\varepsilon}_-\,. \quad (2.16\text{c})$$

The determinant of $\mathbf{C}$ is expressed through eigenvalues in the form:

$$\begin{aligned}\det \mathbf{C} &= C_0\,C_{+1}\,C_{-1} \\ &= (n^2 - \hat{\varepsilon}_0)(n^2 - \hat{\varepsilon}_{+1})(n^2 - \hat{\varepsilon}_{-1}) \\ &= (n^2 - \hat{\varepsilon}_0)[n^2 - \hat{\varepsilon}_+)^2 - \hat{\varepsilon}_-^2] \quad (2.17) \\ &= (A_0\,A_{+1}\,A_{-1})^{1/2} \\ &= [A_0(A_+^2 - A_-^2)]^{1/2}\,.\end{aligned}$$

Expressing the three-dimensional tensor $\mathbf{A}$ by the "eigenvalue representation"

$$\mathbf{A} = A_0\,{}_0\mathbf{P} + A_+\,2\,\mathrm{Re}\,{}_1\mathbf{P} + A_-\,2i\,\mathrm{Im}\,{}_1\mathbf{P} \quad (2.18)$$

facilitates slightly the explicit calculation of the bilinear form $\boldsymbol{n}\cdot\mathbf{A}\cdot\boldsymbol{n}$. Together with the expression (2.17) for $\det \mathbf{C}$ the dispersion equation (2.1) is formulated in terms of the coefficients A_0, A_+, A_- as

$$n^2\,A_0\cos^2\vartheta + n^2\,A_+\sin^2\vartheta = [A_0(A_+^2 - A_-^2)]^{1/2} \quad (2.19)$$

with ϑ as the angle between the wave normal $\hat{\boldsymbol{n}} = \boldsymbol{n}/n$ and the axial vector $\hat{\boldsymbol{B}}$.

Instead in terms of the coefficients A_0, A_+, A_- one can as well formulate the dispersion equation (2.19) in terms of the eigenvalues of the tensor $\mathbf{C}$. With the relations (2.16) one obtains

$$C_{+1}\,C_{-1}\,n^2\cos^2\vartheta + C_0\,C_+\,n^2\sin^2\vartheta - C_0\,C_{+1}\,C_{-1} = 0 \quad (2.20)$$

and subsequently after some manipulations with (2.15b), (2.16)

$$\tfrac{1}{2}[(n^2 - C_{+1})\,A_{+1} + (n^2 - C_{-1})\,A_{-1}]\sin^2\vartheta + (n^2 - C_0)\,A_0\cos^2\vartheta = 0\,. \quad (2.21)$$

Finally with the relation $n^2 - C_{a'} = \hat{\varepsilon}_{a'}$ (2.15a) Eq. (2.21) can be written as

$$\tfrac{1}{2}[\hat{\varepsilon}_{+1}\,A_{+1} + \hat{\varepsilon}_{-1}\,A_{-1}]\sin^2\vartheta + \hat{\varepsilon}_0\,A_0\cos^2\vartheta = 0 \quad (2.22\text{a})$$

and with the expressions (2.16a) for $A_{a'}$ as

$$\hat{\varepsilon}_0[(n^2 - \hat{\varepsilon}_+)^2 - \hat{\varepsilon}_-^2]\cos^2\vartheta + (n^2 - \hat{\varepsilon}_0)[\hat{\varepsilon}_+(n^2 - \hat{\varepsilon}_+) + \hat{\varepsilon}_-^2]\sin^2\vartheta = 0\,. \quad (2.22\text{b})$$

The form (2.22a) could have been obtained directly by inserting the "diagonal representations" for $\hat{\boldsymbol{\epsilon}}$ (2.6) and $\mathbf{A}$ into the dispersion equation (2.2). For a magnetized plasma Eq. (2.22) is Aström's dispersion equation [6]. The quantities A_0, $A_{\pm 1}$ are given by Eqs. (2.16a), the eigenvalues of the effective dielectric tensor $\hat{\boldsymbol{\epsilon}}$ by (2.12).

For an uniaxial crystal Eq. (2.22) is Fresnel's wave normal equation ([2], Sect. 14.3.2, Eqs. (2), (3)). The eigenvalues of the effective dielectric tensor $\hat{\boldsymbol{\epsilon}}$ are given by (2.14b). The angle ϑ is the angle between the wave normal and the principal axis.

Up to now the dispersion equations (Eqs. (2.19) to (2.22)) have been formulated in terms of the eigenvalues $\hat{\varepsilon}_{a'}$ of the effective dielectric tensor $\hat{\boldsymbol{\epsilon}}$. (The eigenvalues $A_{a'}$ and $C_{a'}$ are related to $\hat{\varepsilon}_{a'}$ in Eqs. (2.15), (2.16)). Sometimes it is more convenient to formulate the dispersion equation in terms of the eigenvalues of the inverse effective dielectric tensor $\hat{\boldsymbol{\epsilon}}^{-1}$. To do this we introduce the tensor

$$\begin{aligned}\mathbf{R} &:= \mathbf{C}\cdot\hat{\boldsymbol{\epsilon}}^{-1}\frac{1}{n^2} = (\hat{\boldsymbol{\epsilon}}\cdot\mathbf{A})^{-1}\frac{1}{n^2}\det\mathbf{C} \\ &= \hat{\boldsymbol{\epsilon}}^{-1} - \frac{1}{n^2}\,\mathbf{I}\,, \end{aligned} \quad (2.23)$$

which differs from the inverse effective dielectric tensor $\hat{\boldsymbol{\epsilon}}^{-1}$ only by $\mathbf{I}/n^2$. Since the orthogonal projectors ${}_{a'}\mathbf{P}$ (2.7) form a complete set the tensor $\mathbf{R}$ can be expressed also with a "diagonal representation" (2.6) or an "eigenvalue representation" (2.10) with the eigenvalues and coefficients related to $\hat{\varepsilon}_{a'}$, $A_{a'}$, $C_{a'}$ through

$$\begin{aligned}R_{a'} &= \frac{C_{a'}}{\hat{\varepsilon}_{a'}}\frac{1}{n^2} \\ &= \frac{1}{\hat{\varepsilon}_{a'}\,A_{a'}}\frac{1}{n^2}[A_0(A_+^2 - A_-^2)]^{1/2} \\ &= \frac{1}{\hat{\varepsilon}_{a'}} - \frac{1}{n^2}\,, \quad a' = 0, \pm 1\,, \end{aligned} \quad (2.24)$$

$$R_+ = \frac{1}{2}(R_{+1} + R_{-1}) = \frac{\hat{\varepsilon}_+}{\hat{\varepsilon}_+^2 - \hat{\varepsilon}_-^2} - \frac{1}{n^2}\,,$$

$$R_- = \frac{1}{2}(R_{+1} - R_{-1}) = -\frac{\hat{\varepsilon}_-}{\hat{\varepsilon}_+^2 - \hat{\varepsilon}_-^2}\,.$$

With the relations

$$\hat{\varepsilon}_{a'} A_{a'} = \frac{1}{n^2 R_{a'}} [A_0(A_+^2 - A_-^2)]^{1/2}$$

one can rewrite the dispersion equation (2.22) after some manipulations in the form

$$(R_+^2 - R_-^2)\cos^2\vartheta + R_+ R_0 \sin^2\vartheta = 0\,. \quad (2.25\,\text{a})$$

Subsequently one obtains

$$R_+^2 - R_+(R_+ - R_0)\sin^2\vartheta - R_-^2\cos^2\vartheta = 0. \quad (2.25\,\text{b})$$

For media without spatial dispersion the eigenvalues of the effective dielectric tensor $\hat{\boldsymbol{\varepsilon}}_{a'}$ and subsequently $(R_+ - R_0)$, R_- (Eq. (2.24)) are independent of the refractive index n. Therefore we solve Eq. (2.25 b) for R_+ and obtain

$$\frac{\hat{\varepsilon}_+}{\hat{\varepsilon}_+^2 - \hat{\varepsilon}_-^2} - \frac{1}{n^2} = R_+ = \frac{\sin^2\vartheta}{2}(R_+ - R_0) \cdot \left[1 \pm \sqrt{1 + \left(\frac{2\cos\vartheta\, R_-}{\sin^2\vartheta\,(R_+ - R_0)}\right)^2}\,\right]. \quad (2.26)$$

Equation (2.26) is Försterling's dispersion equation [7]. It relates the refractive index to the frequency $\omega/2\pi$ and the angle ϑ.

3. Covariant dispersion equations for moving gyrotropic media

In the following we will generalize the three-dimensional calculations of the previous section to four dimensions, i.e. we will derive a covariant dispersion equation for gyrotropic media. To do this we start with the covariant dispersion equations for generally anisotropic media ([1], Eq. (4.14))

$$n_\mu A^\mu{}_\nu n^\nu - \det C = 0 \quad (3.1)$$

and ([1], Eq. (4.16))

$$n_\lambda \hat{\varepsilon}^\lambda{}_\mu A^\mu{}_\nu n^\nu = 0\,, \quad (3.2)$$

respectively, and specialize these equations to gyrotropic media applying the same methods as for the three-dimensional calculations.

The four-dimensional vector n^ν, occurring in Eqs. (3.1), (3.2) is defined by ([1], Eq. (3.9 a))

$$n^\nu := \frac{1}{-k_\gamma u^\gamma}(\delta^\nu_\lambda + u^\nu u_\lambda)k^\lambda = \frac{k^\nu}{-k_\gamma u^\gamma} - u^\nu$$
$$= \frac{\gamma}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\left[\boldsymbol{v}\cdot\left(\boldsymbol{k} - \frac{\omega}{c^2}\boldsymbol{v}\right) \,\vdots\, c\left(\boldsymbol{k} - \frac{\omega}{c^2}\boldsymbol{v}\right) - \frac{v^2}{c}\left(\mathbf{1} - \frac{\boldsymbol{v}\,\boldsymbol{v}}{v^2}\right)\cdot\boldsymbol{k}\right] \quad (3.3\,\text{a})$$

with the normalized four-velocity

$$u^\nu := \gamma[1 \,\vdots\, \boldsymbol{v}/c], \quad u^\nu u_\nu = -1\,,$$
$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad (3.3\,\text{b})$$

The four-dimensional tensor $C^\nu{}_\lambda$ and its adjoint $A^\nu{}_\lambda$, defined ([1], Eq. (4.11)) analogous to their three-dimensional counterparts (2.2), (2.4), reduce under the same restrictions as in the comoving frame (i.e. the restriction to media electrically anisotropic but magnetically isotropic in the comoving frame) to, ([1], Eqs. (5.10), (5.11))

$$C^\nu{}_\lambda = n_\gamma n^\gamma \delta^\nu_\lambda - \hat{\varepsilon}^\nu{}_\lambda\,, \quad (3.4\,\text{a})$$

$$C^\mu{}_\nu A^\nu{}_\lambda = A^\mu{}_\nu C^\nu{}_\lambda = \delta^\mu_\lambda \det C\,. \quad (3.4\,\text{b})$$

The four-dimensional effective dielectric tensor $\hat{\varepsilon}^\nu{}_\lambda$ is defined by ([1], Eq. (3.12))

$$\hat{\varepsilon}^\nu{}_\lambda := \frac{1}{\varepsilon_0 \mu_0 \varkappa}\left(\varepsilon^\nu{}_\lambda - \frac{i}{c\,k_\gamma u^\gamma}\sigma^\nu{}_\lambda\right). \quad (3.5)$$

In analogy to the three-dimensional calculations we look for a representation of the effective dielectric tensor by means of projectors in order to simplify the explicit calculation of the quadratic forms in the dispersion Eqs. (3.1), (3.2). In the case of a moving gyrotropic medium the mixed space-time components of $\hat{\varepsilon}^\nu{}_\lambda$ vanish in the comoving frame and the space components are given there by a three-dimensional axial tensor. Thus $\hat{\varepsilon}^\nu{}_\lambda$ can be represented with the complete set of (hermitian) orthogonal four-dimensional projectors ([3], Eq. (B.4))

$${}_{\pm1}P^\nu{}_\lambda := \tfrac{1}{2}(\delta^\nu_\lambda - b^\nu b_\lambda + u^\nu u_\lambda \pm i\,\varepsilon^{\gamma\nu}{}_{\delta\lambda} b^\delta u_\gamma)$$
$$= {}_{\pm1}P^{*\nu}{}_\lambda\,,$$
$${}_0P^\nu_\lambda := b^\nu b_\lambda\,, \quad {}_2P^\nu_\lambda := -u^\nu u_\lambda \quad (3.6)$$

in the "diagonal form" ([3], Eq. (B.9))

$$\hat{\varepsilon}^\nu{}_\lambda = \hat{\varepsilon}_{+1}\,{}_{+1}P^\nu{}_\lambda + \hat{\varepsilon}_0\,{}_0P^\nu_\lambda + \hat{\varepsilon}_{-1}\,{}_{-1}P^\nu{}_\lambda + \hat{\varepsilon}_2\,{}_2P^\nu_\lambda \quad (3.7)$$

with u^ν as the normalized four-velocity ($u^\nu u_\nu = -1$) and $\varepsilon^{\gamma\nu}{}_{\delta\lambda}$ the four-dimensional Levi-Civity symbol. In the comoving frame the space components of the four-vector b^ν are the three-dimensional unit vector $\hat{\mathbf{B}}'$, while the time component vanishes there. In the observer's frame the components of the vector b^ν are calculated via a Lorentz-transformation. One obtains

$$b^\nu = \left[\gamma \frac{\boldsymbol{v}}{c} \cdot \hat{\boldsymbol{B}}' \,\vdots\, \left[\mathbf{I} + (\gamma - 1)\frac{\boldsymbol{v}\,\boldsymbol{v}}{v^2}\right] \cdot \hat{\boldsymbol{B}}'\right],$$
$$b^\nu b_\nu = 1, \quad b^\nu u_\nu = 0. \tag{3.8}$$

In addition to the diagonal form (3.7) the four-dimensional effective dielectric tensor $\hat{\varepsilon}^\nu{}_\lambda$ can be expressed by the "eigenvalue representation"

$$\hat{\varepsilon}^\nu{}_\lambda = \hat{\varepsilon}_0\, {}_0P^\nu_\lambda + \hat{\varepsilon}_+ 2\,\mathrm{Re}\, {}_1P^\nu{}_\lambda + \hat{\varepsilon}_- 2i\,\mathrm{Im}\, {}_1P^\nu{}_\lambda + \hat{\varepsilon}_2\, {}_2P^\nu_\lambda, \tag{3.9}$$

with

$$\hat{\varepsilon}_\pm := \tfrac{1}{2}(\hat{\varepsilon}_{+1} \pm \hat{\varepsilon}_{-1}) \tag{3.10}$$

as the coefficients of the (hermitian) tensors

$$_{+1}P^\nu{}_\lambda + {}_{-1}P^\nu{}_\lambda = 2\,\mathrm{Re}\, {}_1P^\nu{}_\lambda = \delta^\nu_\lambda - b^\nu b_\lambda + u^\nu u_\lambda,$$
$$_{+1}P^\nu{}_\lambda - {}_{-1}P^\nu{}_\lambda = 2i\,\mathrm{Im}\, {}_1P^\nu{}_\lambda = i\,\varepsilon^{\gamma\nu}{}_{\delta\lambda}\, b^\delta u_\gamma. \tag{3.11}$$

The moving gyrotropic medium is completely characterized by the four projectors $_{\alpha'}P^\nu{}_\lambda$ ($\alpha' = 0, \pm 1, 2$) and the three eigenvalues $\hat{\varepsilon}_0$, $\hat{\varepsilon}_{\pm 1}$. The fourth eigenvalue $\hat{\varepsilon}_2$, the coefficient of the projector $_2P^\nu_\lambda := -u^\nu u_\lambda$ is arbitrary ([8], Eqs. (2.9), (2.11), (4.12), (4.13)). With the "diagonal representation" (3.7) the definition of the four-dimensional effective dielectric tensor (3.5) can be written as relations between eigenvalues:

$$\hat{\varepsilon}_{\alpha'} = \frac{1}{\varepsilon_0 \mu_0 \varkappa}\left(\varepsilon_{\alpha'} - \frac{i}{c\,k_\gamma u^\gamma}\,\sigma_{\alpha'}\right), \quad \alpha' = 0, \pm 1, 2. \tag{3.12}$$

Because of the Lorentz invariance of the eigenvalues the relations (3.12) could also be obtained formally by replacing ω in relations (2.11) by $-c k_\gamma u^\gamma$. This replacement is now used to generalize the three-dimensional relations (2.12)—(2.14) to four dimensions. In this way one obtains from Eq. (2.12) the covariant representation of $\hat{\varepsilon}_{\alpha'}$ by the eigenvalues of the conductivity tensor for a magnetized plasma ($\varepsilon^\nu{}_\lambda = \varepsilon_0\,\delta^\nu_\lambda$, $\varkappa = 1/\mu_0$)

$$\hat{\varepsilon}_{\alpha'} = 1 - i\,\sigma_{\alpha'}/\varepsilon_0\, c\, k_\gamma u^\gamma, \quad \alpha' = 0, \pm 1, 2. \tag{3.13}$$

For a cold electron plasma the eigenvalues of the conductivity tensor are given by (cf. Eq. (2.13))

$$\sigma_{a'} = \frac{\omega_\mathrm{p}^2\, \varepsilon_0}{i\,c\,k_\gamma u^\gamma + \nu' + i\,a'\,\omega_\mathrm{B}'}, \quad a' = 0, \pm 1 \tag{3.14}$$

with the dashed quantities ν', ω_B' measured in the comoving frame.

For a dielectric crystal ($\sigma^\nu{}_\lambda \equiv 0$) the covariant relation between $\hat{\varepsilon}_{\alpha'}$ and $\varepsilon_{\alpha'}$ is obtained from Eq. (2.14b) in the form

$$\hat{\varepsilon}_{\alpha'} = \varepsilon_{\alpha'}/\varepsilon_0\,\mu_0\,\varkappa, \quad \alpha' = 0, \pm 1, 2. \tag{3.15}$$

Since the tensors $\hat{\varepsilon}^\nu{}_\lambda$ and $C^\nu{}_\lambda$ differ only by a tensor proportional to the four-dimensional unit tensor δ^ν_λ (Eq. (3.4a)), the tensors $C^\nu{}_\lambda$ and its adjoint $A^\nu{}_\lambda$ can also be expressed by a "diagonal representation" (according to (3.7)) and an "eigenvalue representation" (according to (3.9)), respectively. These representations of $C^\nu{}_\lambda$ and $A^\nu{}_\lambda$ together with the corresponding representations of $\hat{\varepsilon}^\nu{}_\lambda$ allow to write the definition of $C^\nu{}_\lambda$ and $A^\nu{}_\lambda$ (Eqs. (3.4)) as relations between eigenvalues. One obtains, analogous to (2.15), (2.16),

$$C_{\alpha'} = n_\gamma n^\gamma - \hat{\varepsilon}_{\alpha'}, \quad \alpha' = 0, \pm 1, 2,$$
$$C_+ = \tfrac{1}{2}(C_{+1} + C_{-1}) = n_\gamma n^\gamma - \hat{\varepsilon}_+,$$
$$C_- = \tfrac{1}{2}(C_{+1} - C_{-1}) = -\hat{\varepsilon}_-, \tag{3.16}$$

and

$$A_{\alpha'} = C_{\beta'}\, C_{\gamma'}\, C_{\delta'}$$
$$= (n_\gamma n^\gamma - \hat{\varepsilon}_{\beta'})(n_\gamma n^\gamma - \hat{\varepsilon}_{\gamma'})(n_\gamma n^\gamma - \hat{\varepsilon}_{\delta'}),$$
$$\alpha', \beta', \gamma', \delta' = 0, \pm 1, 2; \quad \alpha' \neq \beta' \neq \gamma' \neq \delta',$$
$$A_+ = \tfrac{1}{2}(A_{+1} + A_{-1}) = C_0\, C_2\, C_+$$
$$= (n_\gamma n^\gamma - \hat{\varepsilon}_0)(n_\gamma n^\gamma - \hat{\varepsilon}_2)(n_\gamma n^\gamma - \hat{\varepsilon}_+),$$
$$A_- = \tfrac{1}{2}(A_{+1} - A_{-1}) = -C_0\, C_2\, C_-$$
$$= (n_\gamma n^\gamma - \hat{\varepsilon}_0)(n_\gamma n^\gamma - \hat{\varepsilon}_2)\,\hat{\varepsilon}_-. \tag{3.17}$$

The determinant of $C^\nu{}_\lambda$ is expressed by $\hat{\varepsilon}_{\alpha'}$, $C_{\alpha'}$, $A_{\alpha'}$ in the form

$$\det C = C_0\, C_{+1}\, C_{-1}\, C_2$$
$$= (n_\gamma n^\gamma - \hat{\varepsilon}_0)(n_\gamma n^\gamma - \hat{\varepsilon}_{+1})$$
$$\cdot (n_\gamma n^\gamma - \hat{\varepsilon}_{-1})(n_\gamma n^\gamma - \hat{\varepsilon}_2)$$
$$= (n_\gamma n^\gamma - \hat{\varepsilon}_0)(n_\gamma n^\gamma - \hat{\varepsilon}_2)[(n_\gamma n^\gamma - \hat{\varepsilon}_+)^2 - \hat{\varepsilon}_-^2]$$
$$= (A_0\, A_{+1}\, A_{-1}\, A_2)^{1/3}$$
$$= [A_0\, A_2 (A_+^2 - A_-^2)]^{1/3}. \tag{3.18}$$

Expressing the four-dimensional tensor $A^\mu{}_\nu$ through an "eigenvalue representation", analogous to Eq. (3.9), and inserting this representation into the dispersion equation (3.1) we obtain together with Eqs. (3.6), (3.11), (3.18)

$$A_0 (n_\gamma b^\gamma)^2 + A_+ [n_\gamma n^\gamma - (n_\gamma b^\gamma)^2]$$
$$= [A_0\, A_2 (A_+^2 - A_-^2)]^{1/3}. \tag{3.19}$$

This is the covariant form of the dispersion equation (2.19). In deriving Eq. (3.19) the two terms $n_\mu\, \mathrm{Im}\, {}_1P^\mu{}_\nu n^\nu$ and $n_\mu\, {}_2P^\mu_\nu\, n^\nu = (n_\mu u^\mu)^2$ have dropped out, because of the skewsymmetry of $\mathrm{Im}\, {}_1P^\mu{}_\nu$ (3.11) and the orthogonality of the four-vectors n_μ (3.3) and u^μ.

The insertion of the "diagonal representation" (3.7) for $\hat{\varepsilon}^\nu{}_\lambda$ and the corresponding representation for $A^\mu{}_\nu$ into the dispersion equation (3.2) leads to

$$[n_\gamma n^\gamma - (n_\gamma b^\gamma)^2] \cdot \tfrac{1}{2} [\hat{\varepsilon}_{+1} A_{+1} + \hat{\varepsilon}_{-1} A_{-1}] + A_0 \hat{\varepsilon}_0 (n_\gamma b^\gamma)^2 = 0\,, \tag{3.20}$$

where the term $n_\mu\, {}_2P^\mu_\nu\, n^\nu$ has dropped out again. Equation (3.20) is the generalization of Eq. (2.22) to four dimensions. Thus it is the covariant form of Aström's dispersion equation for magnetized plasma and the covariant form of Fresnel's wave normal equation for an uniaxial crystal.

As discussed in the three-dimensional case it is sometimes more convenient to express the dispersion equation in terms of the eigenvalues of the inverse effective dielectric tensor. To do this we define the four-dimensional tensor

$$R^\nu{}_\mu := C^\nu{}_\lambda (\hat{\varepsilon}^{-1})^\lambda{}_\mu \frac{1}{n_\gamma n^\gamma} = (\hat{\varepsilon}^{-1})^\nu{}_\mu - \frac{1}{n_\gamma n^\gamma} \delta^\nu_\mu, \tag{3.21}$$

which differs from the inverse effective dielectric tensor $(\hat{\varepsilon}^{-1})^\nu{}_\mu$ only by $\delta^\nu_\mu / n_\gamma n^\gamma$. Consequently it can be expressed by a "diagonal" and an "eigenvalue representation", which allows to write the definition (3.21) as a relation between eigenvalues:

$$R_{\alpha'} = \frac{C_{\alpha'}}{\hat{\varepsilon}_{\alpha'} n_\gamma n^\gamma} = \frac{1}{\hat{\varepsilon}_{\alpha'} A_{\alpha'}} \frac{1}{n_\gamma n^\gamma} \cdot [A_0 A_2 (A_+^2 - A_-^2)]^{1/3} = \frac{1}{\hat{\varepsilon}_{\alpha'}} - \frac{1}{n_\gamma n^\gamma}\,, \quad \alpha' = 0, \pm 1, 2\,,$$

$$R_+ = \frac{1}{2}(R_{+1} + R_{-1}) = \frac{\hat{\varepsilon}_+}{\hat{\varepsilon}_+^2 - \hat{\varepsilon}_-^2} - \frac{1}{n_\gamma n^\gamma}\,,$$
$$R_- = \frac{1}{2}(R_{+1} - R_{-1}) = -\frac{\hat{\varepsilon}_-}{\hat{\varepsilon}_+^2 - \hat{\varepsilon}_-^2}\,. \tag{3.22}$$

Expressing $\hat{\varepsilon}_{\alpha'} A_{\alpha'}$ as

$$\hat{\varepsilon}_{\alpha'} A_{\alpha'} = \frac{1}{R_{\alpha'}} \frac{1}{n_\gamma n^\gamma} [A_0 A_2 (A_+^2 - A_-^2)]^{1/3}$$

and inserting it into Eq. (3.20), one obtains

$$(R_+^2 - R_-^2)(n_\gamma b^\gamma)^2 + R_0 R_+ [n_\gamma n^\gamma - (n_\gamma b^\gamma)^2] = 0\,. \tag{3.23}$$

The same manipulations, which in the previous section have led from Eq. (2.25a) to Eq. (2.26), now lead to

$$\frac{\hat{\varepsilon}_+}{\hat{\varepsilon}_+^2 - \hat{\varepsilon}_-^2} - \frac{1}{n_\gamma n^\gamma} = R_+ = \frac{1 - (\hat{n}_\lambda b^\lambda)^2}{2} (R_+ - R_0) \cdot \left[1 \pm \sqrt{1 + \left[\frac{2 \hat{n}_\gamma b^\gamma R_-}{[1 - (\hat{n}_\gamma b^\gamma)^2](R_+ - R_0)} \right]^2} \right], \tag{3.24}$$

with $\hat{n}_\lambda := n_\lambda / (n_\gamma n^\gamma)^{1/2}$ as the normalized four-vector n_λ (3.3 a). In the comoving frame ($n_\gamma' n'^\gamma = n'^2$, $\hat{n}_\lambda' b^\lambda = \cos\vartheta$) Eq. (3.24) is Försterling's dispersion equation (2.26). As discussed in Sect. 2 the right-hand side of the Försterling dispersion equation (2.26) depends, for time dispersive media, only on the frequency $\omega'/2\pi$ and the angle ϑ. Thus the refractive index n' — occurring only on the left side of Eq. (2.26) — is expressed by the frequency $\omega'/2\pi$ and the angle ϑ. This does not hold in the observer's frame. The eigenvalues of the material tensors (cf. Eqs. (3.12)—(3.15)) for time dispersive media become in the observer's frame dependent on $k_\gamma u^\gamma$. Subsequently $R_+ - R_0$, R_- (3.22) depend on $k_\gamma u^\gamma$, too. Thus the right-hand side of Eq. (3.24) depends on the four-dimensional inner products $k_\gamma u^\gamma$ and $\hat{n}_\lambda b^\lambda$, which are related to the three-dimensional wave vector $\boldsymbol{k}$ via (cf. (3.3), (3.8)):

$$k_\gamma u^\gamma = -\frac{\gamma}{c}(\omega - \boldsymbol{k} \cdot \boldsymbol{v})\,, \tag{3.25 a}$$

$$\hat{n}_\gamma b^\gamma = \frac{\frac{\omega}{c} b^0 - \boldsymbol{k} \cdot \boldsymbol{b}}{\frac{\gamma}{c}(\omega - \boldsymbol{k} \cdot \boldsymbol{v})} \frac{1}{(n_\gamma n^\gamma)^{1/2}} \tag{3.25 b}$$

with

$$n_\gamma n^\gamma = \frac{k^2 - \frac{\omega^2}{c^2} + \frac{\gamma^2}{c^2}(\omega - \boldsymbol{k} \cdot \boldsymbol{v})^2}{\frac{\gamma^2}{c^2}(\omega - \boldsymbol{k} \cdot \boldsymbol{v})^2}\,. \tag{3.25 c}$$

As it is obvious from Eqs. (3.25) the right-hand side of Eq. (3.24) becomes in the observer's frame dependent on $|\boldsymbol{k}|$ and $n = |\boldsymbol{k}| \cdot c/\omega$, respectively. Contrary to the comoving frame, in the observer's frame Eq. (3.24) is no suitable form of the dispersion equation, because it connects the refractive index $n := c\,|\boldsymbol{k}|/\omega$ and the frequency $\omega/2\pi$ in a complicated manner.

Obviously one cannot express the refractive index in the observer's frame by the frequency and one angle only, because two directions are distinguished, that of the magnetic field and that of the velocity. Furthermore the four-dimensional inner products $k_\gamma u^\gamma$ and $n_\gamma b^\gamma$ (3.25) cannot be normalized by the absolute value of the three-dimensional wave vector $|\boldsymbol{k}|$ and expressed through angles only. Thus if one solves the dispersion equation for the refractive index $n = c|\boldsymbol{k}|/\omega$, one cannot express the refractive index through the frequency and two angles. What one can do is to express the refractive index through the frequency and two components of the wave vector, e.g.

$$k_v := \boldsymbol{k}\cdot\boldsymbol{v}/v, \quad k_b := \boldsymbol{k}\cdot\boldsymbol{b}/|\boldsymbol{b}| \,. \tag{3.26}$$

To do this we rearrange Eq. (3.23) in the form

$$\left(\hat{R}_+ - \frac{1}{n_\gamma n^\gamma}\right)(R_+ - R_0)\frac{(n_\gamma b^\gamma)^2}{n_\gamma n^\gamma} + \left(\hat{R}_+ - \frac{1}{n_\gamma n^\gamma}\right)\left(\hat{R}_0 - \frac{1}{n_\gamma n^\gamma}\right) - R_-^2\,\frac{(n_\gamma b^\gamma)^2}{n_\gamma n^\gamma} = 0\,. \tag{3.27}$$

The quantities $\hat{R}_+$, $\hat{R}_0$ are defined by

$$\hat{R}_+ := R_+ + \frac{1}{n_\gamma n^\gamma} = \frac{\hat{\varepsilon}_+}{\hat{\varepsilon}_+^2 - \hat{\varepsilon}_-^2}\,, \qquad \hat{R}_0 := R_0 + \frac{1}{n_\gamma n^\gamma} = \frac{1}{\hat{\varepsilon}_0}\,. \tag{3.28}$$

Now the quantities $R_+ - R_0$, R_-, $\hat{R}_+$, $\hat{R}_0$ ((3.22), (3.28)) do not explicitly contain $n_\gamma n^\gamma$. They depend on the material quantities $\hat{\varepsilon}_{\alpha'}$, which for time dispersive media in the comoving frame depend only on the frequency $\omega/2\pi$ and the component k_v (cf. Eqs. (3.13), (3.14), (3.25a)). On the other hand $n_\gamma b^\gamma$ (3.25b) depends on k_v and k_b. Thus the four-dimensional inner product $n_\gamma n^\gamma$ (3.25c) is the only term in Eq. (3.27), which explicitly contains the refractive index $n = c|\boldsymbol{k}|/\omega$. Therefore we arrange Eq. (3.27) according to powers of $1/n_\gamma n^\gamma$:

$$\left(\frac{1}{n_\gamma n^\gamma}\right)^2[1 - (R_+ - R_0)(n_\gamma b^\gamma)^2] - \frac{1}{n_\gamma n^\gamma}[R_-^2(n_\gamma b^\gamma)^2 + (\hat{R}_+ - \hat{R}_0) - \hat{R}_+(R_+ - R_0)(n_\gamma b^\gamma)^2] + \hat{R}_+\hat{R}_0 = 0\,. \tag{3.29}$$

The coefficients of this quadratic equation depend still on the wave vector $\boldsymbol{k}$, but only via its components k_v, k_b. Thus solving this quadratic equation for $n_\gamma n^\gamma$, the refractive index $n = c|\boldsymbol{k}|/\omega$ is expressed by the two components k_v, k_b and the frequency $\omega/2\pi$.

4. Qualitative discussion of the dispersion equation for moving media

To discuss the covariant dispersion equation for moving gyrotropic media, we first decompose the four-dimensional expressions into three-dimensional ones. Then the dispersion equation is solved for the wavenumber $|\boldsymbol{k}|$ and subsequently discussed for various media. Moreover we will extract from the three-dimensional wave vector $\boldsymbol{k}$ that part, which is perpendicular to the convection $\boldsymbol{v}$, and solve the dispersion equation for this part of the wave vector (a procedure which proves to be convenient in stratified media moving parallel to the stratifications).

First of all we transform the dispersion equation (3.20) into a more suitable form. Inserting the expression (3.17) for $A_{\pm 1}$, A_0 into Eq. (3.20) leads after some manipulations to

$$\hat{\varepsilon}_0[(n_\gamma n^\gamma - \hat{\varepsilon}_+)^2 - \hat{\varepsilon}_-^2](n_\lambda b^\lambda)^2 + (n_\gamma n^\gamma - \hat{\varepsilon}_0)[\hat{\varepsilon}_+(n_\gamma n^\gamma - \hat{\varepsilon}_+) + \hat{\varepsilon}_-^2]\cdot[n_\lambda n^\lambda - (n_\lambda b^\lambda)^2] = 0\,, \tag{4.1}$$

the generalization of (2.22b) to moving media. Furthermore, expressing the four-vector n_γ (3.3a) by the wave vector k_γ and collecting the terms with the factor $k_\beta b^\beta$ yields

$$[k_\alpha k^\alpha + (k_\alpha u^\alpha)^2(1-\hat{\varepsilon}_0)]\cdot\{\hat{\varepsilon}_+[k_\beta k^\beta + (k_\beta u^\beta)^2(1-\hat{\varepsilon}_+)] + \hat{\varepsilon}_-^2(k_\beta u^\beta)^2\} + \{(\hat{\varepsilon}_0 - \hat{\varepsilon}_+)[k_\alpha k^\alpha + (k_\alpha u^\alpha)^2(1-\hat{\varepsilon}_+)]^2 - \hat{\varepsilon}_-^2(k_\alpha u^\alpha)^2\}(k_\beta b^\beta)^2 = 0\,. \tag{4.2}$$

This is the covariant dispersion equation for gyrotropic media in terms of the four-dimensional wave vector k_α. In the case of a plasma ($\hat{\varepsilon}_{\alpha'} = 1 - i\,\sigma_{\alpha'}/\varepsilon_0\, c\, k_\gamma u^\gamma$ (3.13)) Eq. (4.2) coincides with Eq. (22) of [3].

The four-dimensional inner products $k_\gamma u^\gamma$, $k_\gamma k^\gamma$, $k_\gamma b^\gamma$ are related to the three-dimensional wave vector $\boldsymbol{k}$, the velocity $\boldsymbol{v}$ and the unit axial vector in the comoving frame $\hat{\boldsymbol{B}}'$ explicitely through

$$k_\gamma u^\gamma = -\frac{\gamma}{c}(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,, \tag{4.3a}$$

$$k_\gamma k^\gamma = k^2 - \frac{\omega^2}{c^2}, \tag{4.3b}$$

$$k_\gamma b^\gamma = \left(-\gamma \frac{\omega}{c^2} + (\gamma - 1)\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{v^2}\right)\boldsymbol{v}\cdot\hat{\boldsymbol{B}}' + \boldsymbol{k}\cdot\hat{\boldsymbol{B}}'. \tag{4.3c}$$

In deriving expression (4.3c) we have used expression (3.8) for b^γ.

For a dielectric medium without dispersion (neither spatial nor temporal) the material quantities $\hat{\varepsilon}_{\alpha'}$ (3.15) do not depend on the frequency and the wave vector. Insertion of expressions (4.3) into Eq. (4.2) shows, that in this case the dispersion equation (4.2) is a polynomial of fourth degree in the wave number $|\boldsymbol{k}|$. Contrary to the comoving frame, where the dispersion equation is a biquadratic equation in $|\boldsymbol{k}'|$, odd powers $|\boldsymbol{k}|$ and $|\boldsymbol{k}|^3$ occur in the dispersion equation in the observer's frame. This is due to the occurrence of the terms $k_\gamma u^\gamma$ and $k_\gamma b^\gamma$ (4.3a, c) in Eq. (4.2) (i.e. a Doppler shift and a drag of the axial vector $\hat{\boldsymbol{B}}'$). Only in the special case of the convection $\boldsymbol{v}$ perpendicular to both, the axial vector $\hat{\boldsymbol{B}}'$ and the wave vector $\boldsymbol{k}$ ($\boldsymbol{v}\cdot\hat{\boldsymbol{B}}' = 0 = \boldsymbol{k}\cdot\boldsymbol{v}$) the dispersion equation becomes biquadratic in $|\boldsymbol{k}|$ (cf. Eqs. (4.3a, c)). Thus solving the dispersion equation for $|\boldsymbol{k}|$ one obtains in general four different solutions, i.e. modes.

For time dispersive media in the comoving frame (e.g. a cold electron plasma) the material quantities $\hat{\varepsilon}_{\alpha'}$ ((3.13), (3.14)) depend on $k_\gamma u^\gamma$ and consequently on $\boldsymbol{k}$ (cf. Eq. (4.3a)), too. Thus the dispersion equation becomes a polynomial of degree higher than four in $|\boldsymbol{k}|$. Solving it for $|\boldsymbol{k}|$ leads to additional solutions (modes) in the observer's frame compared with the four modes in the comoving frame. Only in the special case of wave propagation perpendicular to the direction of the convection ($\boldsymbol{k}\cdot\boldsymbol{v} = 0$) the dispersion equation reduces to a polynomial of fourth degree, because $\boldsymbol{k}\cdot\boldsymbol{v} = 0$ yields $k_\gamma u^\gamma = -\gamma\omega/c$ (4.3a) and thus the material quantities $\hat{\varepsilon}_{\alpha'}$ ((3.13), (3.14)) become independent of the wave vector $\boldsymbol{k}$.

Up to now we have discussed the dispersion equation for the wave number $|\boldsymbol{k}|$, i.e. we used the dispersion equation in a coordinate system, where one of the coordinate axes is fixed to the wave vector $\boldsymbol{k}$. Under special physical and geometrical conditions, one or two components of the wave vector $\boldsymbol{k}$ can be prescribed. In the following we assume the component in the direction of the convection $\boldsymbol{v}$ to be prescribed. This holds for media and boundaries which do not vary spatially in the direction of the convection $\boldsymbol{v}$ (Snell's law). A special example is a cold electron plasma with plane stratifications. Mostly the convection $\boldsymbol{v}$ lies in the planes of stratification. Then Snell's law requires that the component k_v has the same value in all stratifications. Thus k_v can be prescribed by a boundary condition.

In this case it is convenient to decompose the three-dimensional wave vector into

$$\boldsymbol{k} = \boldsymbol{k}_t + k_v\,\boldsymbol{v}/v \tag{4.4}$$

with

$$\boldsymbol{k}_t := \left(\mathsf{I} - \frac{\boldsymbol{v}\,\boldsymbol{v}}{v^2}\right)\cdot\boldsymbol{k}, \quad k_v := \boldsymbol{k}\cdot\frac{\boldsymbol{v}}{v}, \tag{4.5}$$

and to solve the dispersion equation for $|\boldsymbol{k}_t|$, i.e. treating the dispersion equation in a coordinate system, where one of the coordinate axes is fixed to $\boldsymbol{k}_t$. The expressions (4.3) for $k_\gamma u^\gamma$, $k_\gamma k^\gamma$, $k_\gamma b^\gamma$ become

$$k_\gamma u^\gamma = -(\gamma/c)(\omega - k_v v), \tag{4.6a}$$

$$k_\gamma k^\gamma = k_t^2 + k_v^2 - (\omega^2/c^2), \tag{4.6b}$$

$$k_\gamma b^\gamma = \gamma\left(-\frac{\omega}{c^2} + \frac{k_v}{v}\right)\boldsymbol{v}\cdot\hat{\boldsymbol{B}}' + \boldsymbol{k}_t\cdot\hat{\boldsymbol{B}}'. \tag{4.6c}$$

For a cold electron plasma the material quantities $\hat{\varepsilon}_{\alpha'}$ ((3.13), (3.14)) depend on $k_\gamma u^\gamma$, i.e. on ω and k_v. They are independent of the vector $\boldsymbol{k}_t$. The only terms containing $\boldsymbol{k}_t$ in the dispersion equation (4.2) are $k_\gamma b^\gamma$ and $k_\gamma k^\gamma$. Thus the dispersion equation (4.2) is a polynomial of fourth degree in $|\boldsymbol{k}_t|$. In the comoving frame the dispersion equation, known as Booker's quartic [9], is also a polynomial of fourth degree for the $\boldsymbol{k}$ component normal to the planes of stratification. Solving it for $\boldsymbol{k}_t$ leads to four different solutions (i.e. modes), as in the comoving frame. In the special case of the convection $\boldsymbol{v}$ parallel to the direction of the axial vector $\hat{\boldsymbol{B}}'$ the four-dimensional scalar product $k_\gamma b^\gamma$ (Eq. (4.6c) is independent of $|\boldsymbol{k}_t|$ and subsequently the dispersion equation (4.2) becomes biquadratic in $|\boldsymbol{k}_t|$. The same happens in the case of the convection $\boldsymbol{v}$ perpendicular to the axial vector $\hat{\boldsymbol{B}}'$ ($k_\gamma b^\gamma$ becomes proportional to $|\boldsymbol{k}_t|$, cf. Equation (4.6c)). (In the comoving system this happens also if the plane of incidence is perpendicular to the magnetic meridian plane, i.e. the plane spanned by $\hat{\boldsymbol{B}}'$ and the direction normal to the stratifications.)

A special case of gyrotropic media are uniaxial media ($\hat{\varepsilon}_- = 0$), which will be investigated in the following. The dispersion equation (4.2) becomes factorized:

$$[k_\alpha k^\alpha + (k_\alpha u^\alpha)^2(1 - \hat{\varepsilon}_+)] \cdot \{\hat{\varepsilon}_+ [k_\beta k^\beta + (k_\beta u^\beta)^2(1 - \hat{\varepsilon}_0)] + (\hat{\varepsilon}_0 - \hat{\varepsilon}_+)[k_\beta k^\beta + (k_\beta u^\beta)^2(1 - \hat{\varepsilon}_+)](k_\gamma b^\gamma)^2\} = 0. \tag{4.7}$$

We consider uniaxial crystals. (An uniaxial cold electron plasma, as the limiting case of a cold electron plasma for infinite magnetic field, had been considered in [3].) The material quantities $\hat{\varepsilon}_{\alpha'}$ (Eq. (3.15)) do not depend on the frequency and the wave vector. Inserting the explicit expressions for the four-dimensional scalar product $k_\alpha k^\alpha$, $k_\alpha u^\alpha$, $k_\gamma b^\gamma$ (4.3) enables to solve the dispersion equation for the wavenumber $|\boldsymbol{k}|$. It turns out that both factors of Eq. (4.7) are polynomials of second degree in $|\boldsymbol{k}|$. Contrary to the comoving there are two different ordinary modes (resulting from the vanishing of the first factor in Eq. (4.7)) and two different extraordinary modes. The splitting up into two ordinary modes is due to the occurrence of the four-dimensional scalar product $k_\alpha u^\alpha$ (Doppler Shift). For wave propagation perpendicular to the convection ($k_v = 0$) the two modes differ only by their signs. The splitting up into two extraordinary modes is due to the occurence of $k_\alpha u^\alpha$ and $k_\gamma b^\gamma$ (Doppler shift and drag of the principal axis $\hat{\boldsymbol{B}}'$). The two extraordinary modes degenerate into one (i.e. they differ only by their signs) in the special case of $\boldsymbol{k} \cdot \boldsymbol{v} = 0 = \hat{\boldsymbol{B}}' \cdot \boldsymbol{v}$.

In the even more special case of a medium isotropic in the comoving frame ($\hat{\varepsilon}_0 = \varepsilon_+$, $\hat{\varepsilon}_- = 0$) the dispersion equation (4.2) becomes

$$[k_\alpha k^\alpha + (k_\alpha u^\alpha)^2(1 - \hat{\varepsilon}_0)]^2 = 0. \tag{4.8}$$

This is equal to the factor for the ordinary modes in the dispersive equation (4.7) for uniaxial media.

5. Dispersion equation for special directions of wave propagation

The covariant dispersion equation (4.1) consists of a sum of two terms. For special values of the four-vector n_λ, viz.

$$n_\lambda := -k_\lambda / k_\gamma u^\gamma - u_\lambda \sim b_\lambda \tag{5.1}$$

and

$$n_\lambda b^\lambda \equiv -k_\lambda b^\lambda / k_\gamma u^\gamma = 0 \tag{5.2}$$

one of the two terms in Eq. (4.1) vanishes identically. The vanishing of the other term then leads to the dispersion equation. Thus the conditions (5.1) and (5.2), respectively, yield the dispersion equations

$$\hat{\varepsilon}_0[(n_\gamma n^\gamma - \hat{\varepsilon}_+)^2 - \hat{\varepsilon}_-^2] = \hat{\varepsilon}_0(n_\gamma n^\gamma - \hat{\varepsilon}_{+1})(n_\gamma n^\gamma - \hat{\varepsilon}_{-1}) = 0 \quad \text{for} \quad n_\lambda \sim b_\lambda, \tag{5.3}$$

and

$$(n_\gamma n^\gamma - \hat{\varepsilon}_0)[\hat{\varepsilon}_+(n_\gamma n^\gamma - \hat{\varepsilon}_+) - \hat{\varepsilon}_-^2] = 0 \quad \text{for} \quad n_\lambda b^\lambda = 0. \tag{5.4}$$

To discuss the conditions (5.1), (5.2) we formulate them in terms of the three-dimensional wave vector $\boldsymbol{k}$. Because both four-vectors n^λ, b^λ are perpendicular to the four-velocity u_λ, cf. (3.3a), (3.8), only three of the four Eqs. (5.1) are linearly independent. Thus we omit one equation, e.g. the equation for the time component $\lambda = 0$. This leads, after multiplication with the projector $\mathbf{I} - (\boldsymbol{v}\boldsymbol{v}/v^2)$, to a condition for $\boldsymbol{k}$ in the form

$$\boldsymbol{k} - \frac{\omega}{c}\frac{\boldsymbol{v}}{c} = C\left(\mathbf{I} - \frac{\boldsymbol{v}}{c}\frac{\boldsymbol{v}}{c}\right) \cdot \boldsymbol{b}, \tag{5.5}$$

with C as an arbitrary proportionality constant. The three-dimensional vector $\boldsymbol{b}$ is the space part of b^λ (3.8).

In order to obtain a condition for the three-dimensional wave vector $\boldsymbol{k}$ from Eq. (5.2) one decomposes Eq. (5.2) into space parts $\lambda = l$ and time part $\lambda = 0$. This yields together with the relation $b^0 = (\boldsymbol{v}/c) \cdot \boldsymbol{b}$, which follows from $u_\lambda b^\lambda = 0$,

$$\left(\boldsymbol{k} - \frac{\omega}{c}\frac{\boldsymbol{v}}{c}\right) \cdot \boldsymbol{b} = 0. \tag{5.6}$$

In the comoving frame ($\boldsymbol{v} = 0$, $\boldsymbol{b} = \hat{\boldsymbol{B}}'$) Eqs. (5.5), (5.6) are the conditions for wave propagation along and across the symmetry axis $\hat{\boldsymbol{B}}'$. In the observer's frame Eqs. (5.5), (5.6) impose conditions on the dragged $\boldsymbol{k}$ vector $\boldsymbol{k} - \frac{\omega}{c}\frac{\boldsymbol{v}}{c}$, instead of $\boldsymbol{k}$ only. Contrary to the comoving frame neither the two vectors $\boldsymbol{k}$, defined by Eq. (5.5) and (5.6), respectively, are perpendicular to each other, nor the corresponding dragged vectors $\boldsymbol{k} - \frac{\omega}{c}\frac{\boldsymbol{v}}{c}$. This is due to an additional dragg of the symmetry axis $\boldsymbol{B}'$. The two vectors $\boldsymbol{k}$, defined by Eq. (5.5) and (5.6), respectively, are perpendicular to each other only in the case of $\boldsymbol{v} \cdot \boldsymbol{b} = 0$, $\boldsymbol{k} \cdot \boldsymbol{v} = 0$, where the latter relation holds for the wave vector $\boldsymbol{k}$ defined by Equation (5.6). Nevertheless, in the following we will speak — in the observer's frame, too — of propagation along and across the principal axis according to the behaviour in the comoving frame.

In the following we will consider a moving cold electron plasma and discuss qualitatively the dispersion equations for wave propagation along and across the principal axis (i.e. magnetic field). To do this we first rewrite the results for the cold electron plasma at rest. In the comoving frame ($n_\gamma n^\gamma = n'^2$) the dispersion equation (5.3) for wave propagation along the principal axis becomes

$$\hat{\varepsilon}_0 (n'^2 - \hat{\varepsilon}_{+1})(n'^2 - \hat{\varepsilon}_{-1}) = 0 \quad \text{for} \quad \boldsymbol{n}' \sim \boldsymbol{B}' . \tag{5.7}$$

The eigenvalues $\hat{\varepsilon}_0$, $\hat{\varepsilon}_{\pm 1}$ are given by Eqs. (2.12), (2.13) in the comoving frame. Without collisional damping ($\nu' = 0$) the vanishing of ε_0 yields the plasma oscillations $\omega' = \pm\, \omega_\text{p}$. The vanishing of the other factors in Eq. (5.7) leads to the factorized dispersion equation for the two electromagnetic waves with opposit senses of polarizations, i.e. the square of the refractive index equals the eigenvalues $\hat{\varepsilon}_{\pm 1}$. Thus waves propagating along the principal axis are principle waves (cf. [1], Section 6).

On the other hand the dispersion equation (5.4) becomes in the comoving frame factorized as

$$(n'^2 - \hat{\varepsilon}_0)[\hat{\varepsilon}_+ (n'^2 - \hat{\varepsilon}_+) - \hat{\varepsilon}_-^2] = 0 \qquad \text{for} \quad \boldsymbol{n}' \perp \hat{\boldsymbol{B}}' . \tag{5.8}$$

Waves propagating perpendicular to the principle axis are often also called principle waves [10], although they do not propagate parallel to an eigenvector, but perpendicular to one. Furthermore only the vanishing of the first factor in Eq. (5.8) leads to an equality of n'^2 and an eigenvalue.

Next we will formulate the dispersion equations (5.3), (5.4) in terms of the three-dimensional wave vector $\boldsymbol{k}$. Inserting the expression (3.3a) for n_λ into Eq. (5.3) leads to

$$\hat{\varepsilon}_0 [k_\gamma k^\gamma + (k_\gamma u^\gamma)^2 (1 - \hat{\varepsilon}_{+1})] \cdot [k_\beta k^\beta + (k_\beta u^\beta)^2 (1 - \hat{\varepsilon}_{-1})] = 0 \qquad \text{for} \quad n_\lambda \sim b_\lambda . \tag{5.9}$$

The four-dimensional scalar products $k_\gamma k^\gamma$, $k^\gamma u^\gamma$ are related to the three-dimensional wave vector $\boldsymbol{k}$ through

$$k_\gamma u^\gamma = -\frac{\gamma}{c} (\omega - \boldsymbol{k} \cdot \boldsymbol{v}) , \qquad k_\gamma k^\gamma = |\boldsymbol{k}|^2 - \frac{\omega^2}{c^2} . \tag{5.10}$$

Contrary to the comoving frame the dispersion equation for the electromagnetic waves (second and third factor in Eq. (5.9)) is not given by the equality of the square of the refractive index $n := c\,|\boldsymbol{k}|/\omega$ with the eigenvalues $\hat{\varepsilon}_{\pm 1}$. The reason is the drag of the three-dimensional wave vector $\boldsymbol{k}$, which leads to an odd power $|\boldsymbol{k}|$ in Equation (5.9). Nevertheless at a first glance one could assume that the second and third factor in Eq. (5.9) are polynomials of second degree in $|\boldsymbol{k}|$ (analogous to the comoving frame). But this holds only for nondispersive (constant) eigenvalues (e.g. for moving dielectric crystals). In the case of a moving cold electron plasma the Doppler shift of the frequency causes the eigenvalues $\hat{\varepsilon}_{\alpha'}$ ((3.13), (3.14)) in the observer's frame to be dependent on $|\boldsymbol{k}|$. Thus the second and third factor of Eq. (5.9) become polynomials of higher degree in $|\boldsymbol{k}|$ compared with the comoving frame. Solving Eq. (5.9) for the wave number $|\boldsymbol{k}|$ leads to additional solutions, i.e. modes. Furthermore, because of the Doppler shift of the frequency, the expression for the plasma oscillations (vanishing of $\hat{\varepsilon}_0$ ((3.13), (3.14)) becomes in the observer's frame (cf. [3], Eq. (27))

$$(\omega - \boldsymbol{k} \cdot \boldsymbol{v})/(1 - (v^2/c^2))^{1/2} = \pm\, \omega_\text{p} .$$

With the definition (3.3) for n_λ the dispersion equation for wave propagation across the principal axis (5.4) becomes

$$[k_\gamma k^\gamma + (k_\gamma u^\gamma)^2 (1 - \hat{\varepsilon}_0)] \cdot \{\hat{\varepsilon}_+ [k_\beta k^\beta + (k_\beta u^\beta)^2 (1 - \hat{\varepsilon}_+)]^2 - \hat{\varepsilon}_-^2\} = 0 \qquad \text{for} \quad n_\lambda b^\lambda = 0 . \tag{5.11}$$

The same conditions hold as in the case of propagation along the principal axis. Again for a moving cold electron plasma the two factors of Eq. (5.11) are polynomials of higher degree in $|\boldsymbol{k}|$ compared with the comoving frame. This leads to additional solutions (modes).

As discussed in Sect. 4 the component k_v of the wave vector $\boldsymbol{k}$ in the direction of the convection $\boldsymbol{v}$ can be prescribed for special physical and geometrical conditions. In this case it is convenient to decompose the wave vector, according to (4.4), into a vector $\boldsymbol{k}_t$ perpendicular to the direction of the convection $\boldsymbol{v}$, and the component k_v. Thus for a prescribed k_v, the conditions (5.5), (5.6) impose conditions on $\boldsymbol{k}_t$ instead of $\boldsymbol{k}$. One obtains

$$\boldsymbol{k}_t = \hat{C}\left(\mathbf{I} - \frac{\boldsymbol{v}\,\boldsymbol{v}}{v^2}\right)\cdot\left(\mathbf{I} - \frac{\boldsymbol{v}\,\boldsymbol{v}}{c^2}\right)\cdot\boldsymbol{b} \qquad (5.12)$$

in the case of propagation along the principal axis, and

$$\boldsymbol{k}_t\cdot\boldsymbol{b} = \left(\frac{\omega}{c^2} - \frac{k_v}{v}\right)\boldsymbol{v}\cdot\boldsymbol{b} \qquad (5.13)$$

in the case of propagation across the principal axis. Because we have fixed one component of $\boldsymbol{k}$ (i.e. k_v) the proportionality factor $\hat{C}$ in Eq. (5.12) is no longer arbitrary. It becomes

$$\hat{C} = \gamma^2\left(k_v - \frac{\omega}{c}\frac{v}{c}\right)\Big/\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{b}. \qquad (5.14)$$

Analogous to the comoving frame, where one could solve the dispersion equations (5.7), (5.8) for k'^2, one can now solve the dispersion equations (5.9), (5.11) for $k_t{}^2$. Contrary to k'^2, $k_t{}^2$ is not simply equal to eigenvalues $\hat{\varepsilon}_{a'}$, but is a function of ω, k_v and of the eigenvalues $\hat{\varepsilon}_{a'}$.

6. Polarization relations in gyrotropic media

In a previous paper [1] we derived polarization relations for anisotropic media using bilinear forms. These three-dimensional and four-dimensional polarization relations will now be specialized to gyrotropic media. The three-dimensional calculations will be given at the beginning. The generalization to four dimensions (moving media) is straightforward.

In the case of an anisotropic medium at rest we obtained the three-dimensional polarization relations ([1], Eq. (7.3))

$$E_1 : E_2 : E_L = \boldsymbol{g}_1\cdot\mathbf{A}\cdot\boldsymbol{n} : \boldsymbol{g}_2\cdot\mathbf{A}\cdot\boldsymbol{n} : \frac{1}{n}\det\mathbf{C}. \qquad (6.1)$$

with $\boldsymbol{E}_1 \sim \boldsymbol{g}_1$ and $\boldsymbol{E}_2 \sim \boldsymbol{g}_2$ as transverse components perpendicular to $\boldsymbol{n}$ and $\boldsymbol{E}_L \sim \boldsymbol{n}$ as longitudinal component of the electirc vector $\boldsymbol{E}$.

For magnetically isotropic media (which we are considering) the polarization relations can also be expressed through bilinear forms of $\hat{\boldsymbol{\epsilon}}\cdot\mathbf{A}$ instead of bilinear forms of $\mathbf{A}$ in Eq. (6.1). For these media the tensor $\hat{\boldsymbol{\epsilon}}/n^2$ is a transverse projector $\mathbf{I} - \boldsymbol{n}\boldsymbol{n}/n^2$ (cf. [1], Eq. (3.3a)). Multiplication of the polarization relation ([1], Eq. (4.4)) $\boldsymbol{E} = \mathbf{A}\cdot\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{E}/\det\mathbf{C})$ with $\hat{\boldsymbol{\epsilon}}/n^2$ yields

$$\frac{\hat{\boldsymbol{\epsilon}}}{n^2}\cdot\boldsymbol{E} = \left(\mathbf{I} - \frac{\boldsymbol{n}\,\boldsymbol{n}}{n^2}\right)\cdot\boldsymbol{E} = \frac{\hat{\boldsymbol{\epsilon}}}{n^2}\cdot\boldsymbol{A}\cdot\boldsymbol{n}\frac{\boldsymbol{n}\cdot\boldsymbol{E}}{\det\mathbf{C}}. \qquad (6.2)$$

Subsequent multiplication with the transverse vectors $\boldsymbol{g}_1, \boldsymbol{g}_2$ leads to the polarization relations

$$E_1 : E_2 : E_L = \boldsymbol{g}_1\cdot\hat{\boldsymbol{\epsilon}}\cdot\mathbf{A}\cdot\boldsymbol{n} : \boldsymbol{g}_2\cdot\hat{\boldsymbol{\epsilon}}\cdot\mathbf{A}\cdot\boldsymbol{n} : n\det\mathbf{C}, \qquad (6.3)$$

equivalent to (6.1) for magnetically isotropic media. For gyrotropic media the three-dimensional tensors $\hat{\boldsymbol{\epsilon}}$, $\mathbf{A}$, $\mathbf{C}$ are axial tensors and are explicitly given by Eqs. (2.5), (2.3), (2.4). In analogy to the calculation of the dispersion equations (Sect. 2) we represent the axial tensors $\hat{\boldsymbol{\epsilon}}$, $\mathbf{A}$ by projectors in order to simplify the explicit calculation of the bilinear forms in Eqs. (6.1), (6.3). Thus expressing $\hat{\boldsymbol{\epsilon}}$ and $\mathbf{A}$ with the "eigenvalue representations" (2.10), (2.18) yields

$$\boldsymbol{g}_{A'}\cdot\mathbf{A}\cdot\boldsymbol{n} = (A_0 - A_+)\,\boldsymbol{g}_{A'}\cdot\hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}}\cdot\boldsymbol{n} + i\,A_-\boldsymbol{g}_{A'}\cdot\hat{\boldsymbol{B}}\times\boldsymbol{n}, \quad A' = 1, 2, \quad (6.4\text{a})$$

and

$$\boldsymbol{g}_{A'}\cdot\hat{\boldsymbol{\epsilon}}\cdot\mathbf{A}\cdot\boldsymbol{n} = [\hat{\varepsilon}_0 A_0 - \tfrac{1}{2}(\hat{\varepsilon}_{+1}A_{+1} + \hat{\varepsilon}_{-1}A_{-1})]\,\boldsymbol{g}_{A'}\,\hat{\boldsymbol{B}}\,\hat{\boldsymbol{B}}\cdot\boldsymbol{n} + i\,n^2 A_-\boldsymbol{g}_{A'}\cdot\hat{\boldsymbol{B}}\times\boldsymbol{n}, \quad A' = 1, 2. \quad (6.4\text{b})$$

In deriving Eq. (6.4b) we have used the relation $\frac{1}{2}(\hat{\varepsilon}_{+1}A_{+1} - \hat{\varepsilon}_{-1}A_{-1}) = n^2 A_-$ (2.16).

Next we choose a special coordinate system. The two transverse vectors $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ form together with the wave normal $\boldsymbol{n} := \boldsymbol{n}/n$ an orthogonal set. To fix this coordinate system completely we choose the direction of one of the transverse vectors parallel to $\hat{\boldsymbol{B}}\times\boldsymbol{n}$ i.e.

$$\boldsymbol{g}_2 \sim \hat{\boldsymbol{B}}\times\boldsymbol{n}, \qquad (6.5\text{a})$$

which leads to the following relations with ϑ as angle between $\boldsymbol{n}$ and $\hat{\boldsymbol{B}}$:

$$\hat{\boldsymbol{B}}\cdot\boldsymbol{n} = n\cos\vartheta, \quad \hat{\boldsymbol{B}}\cdot\boldsymbol{g}_1 = -\sin\vartheta,$$
$$\boldsymbol{g}_2\cdot\hat{\boldsymbol{B}}\times\boldsymbol{n} = n\sin\vartheta. \qquad (6.5\text{b})$$

Imposing the condition (6.5a) on $\boldsymbol{g}_2$ yields together with the Eqs. (6.4), (6.5) the following form of the transverse polarization

$$\frac{E_1}{E_2} = i\,\frac{A_0 - A_+}{A_-}\cos\vartheta \qquad (6.6)$$
$$= i\,\frac{\hat{\varepsilon}_0 A_0 - \frac{1}{2}(\hat{\varepsilon}_{+1} A_{+1} + \hat{\varepsilon}_{-1} A_{-1})}{n^2 A_-}\cos\vartheta\,.$$

Using the dispersion equation (2.22a) one can express $\hat{\varepsilon}_0 A_0$ as

$$\hat{\varepsilon}_0 A_0 = -\tfrac{1}{2}(\hat{\varepsilon}_{+1} A_{+1} + \hat{\varepsilon}_{-1} A_{-1})\sin^2\vartheta/\cos^2\vartheta$$

and eliminate it in the polarization relation (6.6). Thus one obtains together with Eqs. (2.16), (2.24) ([5], Eq. (6.24))

$$\frac{E_1}{E_2} = \frac{\frac{1}{2}(\hat{\varepsilon}_{+1} A_{+1} + \hat{\varepsilon}_{-1} A_{-1})}{i\,n^2 A_- \cos\vartheta} \qquad (6.7)$$
$$= \frac{\hat{\varepsilon}_{+1}\,\hat{\varepsilon}_{-1} - n^2\,\hat{\varepsilon}_+}{i\,n^2\,\hat{\varepsilon}_- \cos\vartheta} = \frac{i}{\cos\vartheta}\,\frac{R_+}{R_-}\,.$$

Replacing R_+ by Försterling's dispersion equation (2.26) allows to express the transverse polarization by a single quantity

$$\zeta := \frac{2\cos\vartheta}{\sin^2\vartheta}\,\frac{R_-}{R_+ - R_0} \qquad (6.8)$$

as ([5], Eq. (6.27))

$$\frac{E_1}{E_2} = \frac{i}{\zeta}\,[1 \pm \sqrt{1+\zeta^2}]\,. \qquad (6.9)$$

For media without spatial dispersion (e.g. a cold plasma) ζ does not depend on the wave number $|\boldsymbol{k}|$. It depends only on the frequency $\omega/2\pi$ and the angle ϑ.

The generalization to four-dimensions is straightforward. For media anisotropic in the comoving frame we obtained with $g^\mu_{A'} := [0 \vdots \boldsymbol{g}_{A'}]$ ([1], Eq. (7.4)), the polarization relations ([1], Eq. (7.12))

$$E_1 : E_2 : E_L = g_1^\mu A_\mu{}^\nu n_\nu : g_2^\mu A_\mu{}^\nu n_\nu : \frac{c}{\omega}|\boldsymbol{k}| \qquad (6.10)$$
$$\cdot\,\frac{(k_\alpha u^\alpha)^2 \det C - k_\alpha k^\alpha u^\beta (g_\beta^1 g_1^\mu + g_\beta^2 g_2^\mu) A_\mu{}^\nu n_\nu}{\gamma\,\boldsymbol{k}\cdot\left(\boldsymbol{k} - \dfrac{\omega\,\boldsymbol{v}}{c^2}\right)}\,.$$

The four-dimensional bilinear forms occurring in Eq. (6.10) can most conveniently be calculated by representing the four-dimensional tensor through an "eigenvalue representation" according to Equation (3.9). Thus one obtains

$$g^\mu_{A'} A_\mu{}^\nu n_\nu = (A_0 - A_+)\,g^\mu_{A'}\,b_\mu b^\nu n_\nu + i\,A_-\,\varepsilon_{\varkappa\nu\gamma\delta}\,u^\gamma b^\delta g^\varkappa_{A'} n^\nu\,. \qquad (6.11)$$

Analogous to the three-dimensional calculations we are now choosing a special coordinate system. The generalization of the three-dimensional condition (6.5a) to four dimensions is

$$g_2^\lambda b_\lambda = 0\,. \qquad (6.12)$$

Imposing this condition (6.12) on the four-vector g_2^λ leads to the following expressions for the bilinear forms (6.11)

$$g_1^\mu A_\mu{}^\nu n_\nu = (A_0 - A_+)\,g_1^\mu\,b_\mu b^\nu n_\nu + i\,A_-\,\varepsilon_{\varkappa\nu\gamma\delta}\,u^\gamma b^\delta g_1^\varkappa n^\nu\,, \qquad (6.13)$$
$$g_2^\mu A_\mu{}^\nu n_\nu = i\,A_-\,\varepsilon_{\varkappa\nu\gamma\delta}\,u^\gamma b^\delta g_2^\varkappa n^\nu\,.$$

The four-dimensional Levi-Civita tensors, occurring in expressions (6.13) for the bilinear forms, make the polarization relations (6.10) for moving media very involved. It is not possible to perform the same procedure as for the three-dimensional calculations. But nevertheless one can use the four-dimensional polarization relations for some simple special cases.

The simplest case is that of a vacuum ($\hat{\varepsilon}_- = 0$, $\hat{\varepsilon}_{\pm 1} = \hat{\varepsilon}_+ = \hat{\varepsilon}_0 = 1$). In this case the eigenvalues and coefficients of the tensor $A_\mu{}^\nu$ (3.17) become: $A_0 = A_+ = (n_\gamma n^\gamma - 1)^2 (n_\gamma n^\gamma - \hat{\varepsilon}_2)$, $A_- = 0$. The dispersion equation $k_\alpha k^\alpha = 0 = n_\gamma n^\gamma - 1$ (4.8) requires $A_0 = A_+ = 0$. Thus the bilinear forms (6.13) become zero, too. The transverse polarization E_1/E_2 (6.9) can take arbitrary values. Because $\det C$ (3.18) goes to zero as $(n_\gamma n^\gamma - 1)^3$, $A_0 = A_+$ and subsequently the bilinear forms as $(n_\gamma n^\gamma - 1)^2$ the longitudinal component vanishes.

In a medium isotropic in the comoving frame ($\hat{\varepsilon}_- = 0$, $\hat{\varepsilon}_+ = \hat{\varepsilon}_0 = \hat{\varepsilon}_{\pm 1}$) the coefficients of the tensor $A_\mu{}^\nu$ become: $A_0 = A_+ = (n_\gamma n^\gamma - \hat{\varepsilon}_0)^2 (n_\gamma n^\gamma - \hat{\varepsilon}_2)$, $A_- = 0$. Again the dispersion equation $n_\gamma n^\gamma - \hat{\varepsilon}_0 = 0$ (4.8) requires $A_0 = A_+ = 0$. Consequently the bilinear forms (6.13) are zero, too. The transverse polarization E_1/E_2 can take arbitrary values. Although $\det C$ goes to zero as $(n_\gamma n^\gamma - \hat{\varepsilon}_0)^3$ the longitudinal component becomes unequal to zero. It is proportional to $v_1 E_1 + v_2 E_2$ in the form

$$\frac{c}{\omega}|\boldsymbol{k}|\,k^\alpha u_\alpha (v_1 E_1 + v_2 E_2) \qquad (6.14)$$
$$= \gamma\,\boldsymbol{k}\cdot\left(\boldsymbol{k} - \frac{\omega}{c}\frac{\boldsymbol{v}}{c}\right) E_L\,.$$

In an uniaxial medium ($\hat{\varepsilon}_- = 0$, which leads to $A_- = 0$ (3.17)), one has to distinguish between the ordinary and the extraordinary mode. The dispersion relation for the ordinary mode leads to the same conditions as in the medium isotropic in the comoving frame. For the extraordinary mode the dispersion equation does not lead to a simplification of the polarization relations. One has to calculate them explicitly by the relations (6.10).

7. Concluding remarks

In this paper dispersion equations and polarization relations for media electrically gyrotropic and magnetically isotropic (in the comoving frame) have been calculated. By a suitable change of the variables the results of Sects. 2, 3, 6 can as well be used for media electrically isotropic and magnetically gyrotropic (in the comoving frame), but not for media which are both, electrically and magnetically, gyrotropic. In this case the material tensors and the tensors which build up the quadratic and bilinear forms, have different symmetry axes (even if the dielectric tensor and the permeability tensor have the same symmetry axis). Thus the explicit calculation of the quadratic and bilinear forms, derived by Hebenstreit and Suchy [1], become more complicated.

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